# Hale School

## Mathematics Specialist

## Test 5 --- Term 3 2019

## Applications of Differentiation and Modelling Motion

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**Instructions:**

- **Calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

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Use exact values in your answers.

**Question 1** **(3, 4 = 7 marks)**

(a) Determine $\frac{dy}{dx}$ for the relation $y\ln(x) = e^{2y} + 3x - 4$.

$$\frac{dy}{dx}\ln x + \frac{y}{x} = 2\frac{dy}{dx}e^{2y} + 3$$

$$\frac{dy}{dx}\left(\ln x - 2e^{2y}\right) = 3 - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{3 - \frac{y}{x}}{\ln x - 2e^{2y}}$$

$$\frac{dy}{dx} = \frac{3x - y}{x\left(\ln x - 2e^{2y}\right)}$$

(b) Find the gradient of the curve with equation $2x^2\sin(y) + xy = \frac{\pi^2}{18}$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$.

Give your answer in the form $\frac{a}{\pi\sqrt{b} + c}$, where $a$, $b$ and $c$ are integers.

$$4x\sin y + 2x^2\cos y\frac{dy}{dx} + y + \frac{dy}{dx}x = 0$$

$$4\frac{\pi}{6}\sin\frac{\pi}{6} + 2\left(\frac{\pi}{6}\right)^2\cos\frac{\pi}{6}\frac{dy}{dx} + \frac{\pi}{6} + \frac{dy}{dx}\frac{\pi}{6} = 0 \quad \sqrt{\text{sub}}$$

$$2 + 2\frac{\pi}{6}\frac{\sqrt{3}}{2}\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\sqrt{3}\frac{\pi}{6} + 1\right) = -3$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{3}\frac{\pi}{6} + 1}$$

$$= \frac{-18}{\pi\sqrt{3} + 6} \quad \sqrt{\text{corr}}$$

**Question 2** **(4 marks)**

Given the stated conditions, determine the general solution to the following differential equation:

$\frac{dy}{dx} = \frac{3-y}{2}$, $y \geq 2$ 3

$$\frac{dy}{dx} = \frac{3-y}{2}$$

$$\int \frac{2}{3-y}\,dy = \int dx$$ ✓ separates variables

$$-2\ln|3-y| = x + c$$ ✓ integrates

$$\ln|3-y| = -\frac{x}{2} + c$$

$$3-y = \pm e^{-\frac{x}{2}+c}$$ ✓

$$y = 3 \pm e^{-\frac{x}{2}+c}$$

$$y = 3 + e^{-\frac{x}{2}+c}$$ $(y \geq 3)$

✓ correct solution for condition

**Question 3** **(4 marks)**

The acceleration of a beam of light along a straight lamp post is given by the expression $a(t) = x - 7$ where $x(t)$ is in metres and $t$ is in seconds, $v(0) = 7$ m/s , $x(0) = 0$ .

Find $v$ in terms of $x$ .

$$a = x - 7$$

$$v\frac{dv}{dx} = x - 7$$ (✓ $v\frac{dv}{dx}$)

$$\int v\,dv = \int x - 7\,dx$$ (✓ integrates)

$$\frac{v^2}{2} = \frac{x^2}{2} - 7x + c$$ $t=0, x=0, v=7$

$$\frac{49}{2} = c$$ ✓ (determine c)

$$\frac{v^2}{2} = \frac{x^2}{2} - 7x + \frac{49}{2}$$

$$v^2 = x^2 - 14x + 49$$

$$v = \cancel{\pm}\sqrt{x^2 - 14x + 49}$$ — ✓ (v in terms of x)

$$v = \cancel{\pm}|x-7|$$

$$v = \sqrt{x^2 - 14x + 49}$$

$$v = |x-7|$$

**Question 4** **(6 marks)**

Use the separation of variables method to solve the following differential equation

$$\frac{dP}{dt} = 0.1P(1-0.05P)\ ,\ P(0)=1\ .$$

✓ separation

$$\int \frac{1}{P(1-0.05P)}\,dP = \int 0.1\,dt$$

$$\frac{1}{P(1-0.05P)} = \frac{A}{P} + \frac{B}{1-0.05P}$$

$$1 = A(1-0.05P) + BP$$

let $P=0$ $\quad A=1$

let $P=20$ $\quad 1 = 20B$

$$B = \frac{1}{20} = 0.05$$

✓ partial fractions

$$\therefore \int \frac{1}{P} + \frac{0.05}{1-0.05P}\,dP = \int 0.1\,dt$$

$$\ln|P| - \ln|1-0.05P| = 0.1t + c$$

✓ integrates

$$\ln\left|\frac{P}{1-0.05P}\right| = 0.1t + c$$

$$\frac{P}{1-0.05P} = e^{0.1t+c}$$

$(P(0)=1)$

$$\frac{1}{0.95} = e^{c}$$

✓ solves for c

$$\therefore \frac{P}{1-0.05P} = \frac{1}{0.95}e^{0.1t}$$

$$P = \frac{1}{0.95}e^{0.1t}(1-0.05P)$$

✓ rearranges (makes P subject)

$$P\left(1 + \frac{0.05}{0.95}e^{0.1t}\right) = \frac{1}{0.95}e^{0.1t}$$

$$P = \frac{\frac{1}{0.95}e^{0.1t}}{1 + \frac{0.05}{0.95}e^{0.1t}}$$

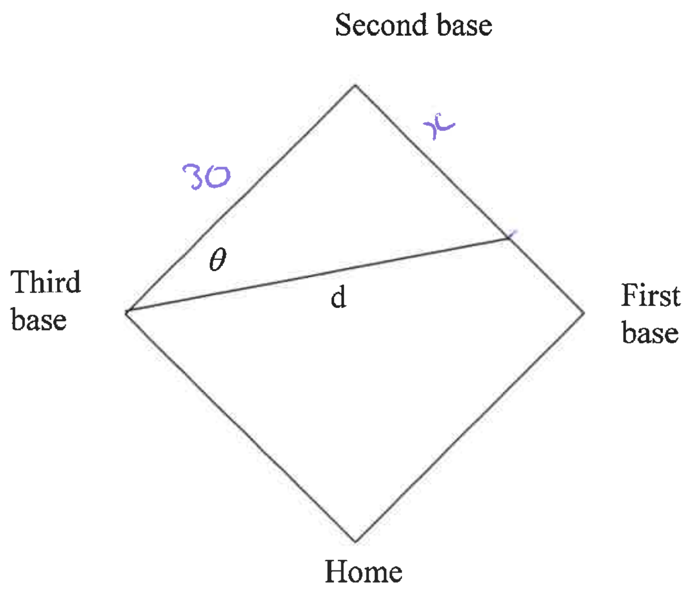
$$P = \frac{1}{0.95e^{-0.1t} + 0.05}$$

$$P = \frac{20}{19e^{-0.1t} + 1}$$

✓ simplifies (removes fractions & decimals)

**Question 5** **(4, 4 = 8 marks)**

A baseball diamond is a square approximately 30 metres on each side. A player runs from first base to second base at a rate of 5 metres per second.




(a) At what rate is the player's distance from third base, d, changing when the player is 10 metres from second base?

formula ✓ $x^2 + 30^2 = d^2$ $\frac{dx}{dt} = -5$ m/s

diff ✓ $2x \frac{dx}{dt} = 2d \frac{dd}{dt}$

$\frac{dd}{dt} = \frac{x \frac{dx}{dt}}{d}$

d ✓ $= \frac{10 \times 5}{\sqrt{10^2+30^2}}$ $-\frac{5}{\sqrt{10}}$ or $\frac{\sqrt{10}}{2}$

$\frac{dd}{dt}$ ✓ (decreasing) $= \frac{50}{10\sqrt{10}} = -\cancel{\frac{1}{2}}$ m/s. decreasing by $\frac{1}{2}$ m/s

(b) As the player slides into second base, the angle θ is changing at 8 degrees per second. Determine the speed of the player in metres per second at this instance.

$\tan\theta = \frac{x}{30}$ ✓ equation

$\frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{30}$ ✓ diff $\theta \Rightarrow 0$ $\frac{d\theta}{dt} = 8 \times \frac{\pi}{180}$

$\frac{1}{1} \cdot \frac{8\pi}{180} = \frac{dx}{dt} \cdot \frac{1}{30}$ ✓ converts to rad.

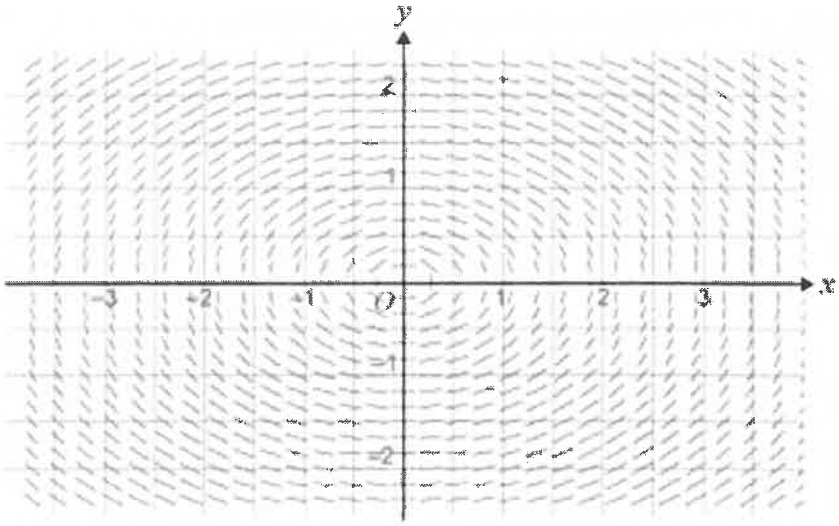
$\frac{4\pi}{3} = \frac{dx}{dt}$

m/s

✓ solution.

**Question 6** **(2, 2 = 4 marks)**

(a) The direction (slope) field for a certain first order differential equation is shown below.




Circle ~~the~~ the differential equation that may respresent the slope field.

A. $\frac{dy}{dx} = \frac{x^2}{2} + y^2$

B. $\frac{dy}{dx} = x^2 + \frac{y^2}{2}$

C. $\frac{dy}{dx} = -\frac{x}{2y}$ ✓✓ correct solution

D. $\frac{dy}{dx} = -\frac{y}{2x}$

E. $\frac{dy}{dx} = \frac{x}{2y}$

(b) For the differential equation $\frac{dy}{dx} = \frac{x}{2y}$ passing through the point (-1,1), use the incremental formula $\delta y = \frac{dy}{dx} \times \delta x$ , with $\delta x = 0.1$ to calculate an estimate for the $y$ - coordinate of the curve when $x = -1.1$ .

$y_1 = y_0 + \delta y$

$y_1 = 1 + \frac{-1}{2(1)} \times (-0.1)$ ✓ incremental formula

$y_1 = 1 + 0.05$

$y_1 = 1.05$ ✓ solution.

$\therefore (-1.1, 1.05)$

**Question 7** **(4, 5 = 9 marks)**

A particle is undergoing Simple Harmonic Motion such that $\dfrac{d^2x}{dt^2} + 4\pi^2 x = 0$

(a) Given that the particle begins at the origin, with positive velocity, and has a maximum velocity of $8\pi$ m/sec, determine the displacement of the particle at any time $t$.

$\dfrac{d^2x}{dt^2} = -4\pi^2 x$ $\therefore$ let $x = A\sin(nt)$

$x = A\sin(2\pi t)$

$V_{max} = nA$

$8\pi = 2\pi A$

$A = 4$

$\therefore x = 4\sin(2\pi t)$

✓ sint function

✓ uses v max

✓ identifies n

✓ solution positive (positive velocity)

(b) Determine the acceleration of the particle when the particle first has negative displacement and a velocity of $4\pi$ m/sec.

First has negative displacement after $t = \frac{1}{2}$ secs ✓ (negative displacement condition)

$T = \left(\frac{2\pi}{n}\right) / \frac{1}{2}$

$= \frac{2\pi}{2\pi} \cdot \frac{1}{2}$

$= \frac{1}{2}$ secs.

$V = 8\pi\cos(2\pi t)$ ✓ (velocity)

$4\pi = 8\pi\cos(2\pi t)$

$\frac{1}{2} = \cos(2\pi t)$

$\frac{5\pi}{3}, \frac{\pi}{3} = 2\pi t$

$t = \frac{5}{6}$ secs ✓ (solves for t)

$x = 4\sin\frac{10\pi}{6} = 4\sin\frac{5\pi}{3}$

$= 4 \cdot -\frac{\sqrt{3}}{2}$

$= -2\sqrt{3}$ ✓ (determines x)

$A = -4\pi^2(-2\sqrt{3})$

$A = 8\sqrt{3}\pi^2$ m/sec$^2$ ✓ (finds a)

__End of Test__